Non-polynomial Fourth Order Equations which Pass the Painlevé Test

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The singular point analysis of fourth order ordinary differential equations in the non-polynomial class are presented. Some new fourth order ordinary differential equations which pass the Painlevé test as well as the known ones are found. — PACS: 02.30.Hq, 02.30.Ik, 02.30.Gp

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1. Introduction

Painlevé and his school [1 – 3] studied the certain class of second order ordinary differential equations (ODEs) and found fifty canonical equations whose solutions have no movable critical points. This property is known as the Painlevé property. Distinguished among these fifty equations are six Painlevé equations, PI – PVI. The six Painlevé transcendents are regarded as nonlinear special functions.

The fourth order equations of Painlevé type

$$y^{(4)} = F(z, y, y', y'', y'''), \tag{1.1}$$

where F is polynomial in y and its derivatives, were considered in [4 – 12]. Third order polynomial type equations with Painlevé property were investigated in [4 – 6, 11, 13]. Non-polynomial third order equations of Painlevé type were studied in [14 – 16]. Some of the fourth order non-polynomial equations possing Painlevé property were introduced in [10, 17 – 21].

In this article, we consider the simplified equation associated with the following fourth order differential equation

$$y^{(4)} = a_1\frac{y'y'''}{y} + a_2\frac{(y'')^2}{y} + a_3\frac{(y')^2y''}{y^2} + a_4\frac{(y')^4}{y^3} + F(z, y, y', y'', y'''), \tag{1.2}$$

where $a_j (j = 1, \ldots, 4)$ are constants not all zero. F may contain the leading terms, but all the terms of F are of order ϵ^{-3} or greater if we let $z = \zeta_0 + \epsilon t$, where ζ_0 is a constant, ϵ a small parameter and t the new independent variable, and the coefficients in F are locally analytic functions of z. The equation of type (1.2) can be obtained by differentiating twice the leading terms of the third (or fourth) Painlevé equation and adding the terms of order -5 or greater as $z \rightarrow z_0$ (i. e. in the neighborhood of the movable pole z_0) with analytic coefficients in z such that:

i) $y = 0, \infty$ are the only singular values of equation in y.
ii) The additional terms are of order ϵ^{-4} or greater, if one lets $z = \zeta_0 + \epsilon t$.

If we let, $z = \zeta_0 + \epsilon t$ and take the limit as $\epsilon \rightarrow 0$, (1.2) yields the following "reduced" equation:

$$\ddddot{y} = a_1\frac{\dot{y}\,\dddot{y}}{y} + a_2\frac{(\ddot{y})^2}{y} + a_3\frac{(\dot{y})^2\ddot{y}}{y^2} + a_4\frac{(\dot{y})^4}{y^3}, \tag{1.3}$$

where $\dot{} = \mathrm{d}/\mathrm{d}t$. Substituting $y \cong y_0(t - t_0)^\alpha$ into equation (1.3) gives

$$\begin{aligned}&(a_1 + a_2 + a_3 + a_4 - 1)\alpha^3\\ &- (3a_1 + 2a_2 + a_3 - 6)\alpha^2\\ &+ (2a_1 + a_2 - 11)\alpha + 6 = 0.\end{aligned} \tag{1.4}$$

Depending on the coefficients of (1.4), we have the following three cases. In the case of single branch, let

$$\begin{aligned}&a_1 + a_2 + a_3 + a_4 - 1 = 0,\\ &3a_1 + 2a_2 + a_3 - 6 = 0,\\ &2a_1 + a_2 - 11 \neq 0,\end{aligned} \tag{1.5}$$

and the root of (1.4) be $\alpha = n \in \mathbb{Z} - \{0\}$. Substituting

$$y \cong y_0(t-t_0)^{\alpha} + \beta(t-t_0)^{r+\alpha}, \tag{1.6}$$

into (1.3), we obtain the following equation for the Fuchs indices:

$$r(r+1)\{r^2 - [(a_1-4)n+7]r+6\} = 0. \tag{1.7}$$

So, the Fuchs indices are $r_0 = -1$, $r_1 = 0$, r_2 and r_3 such that

$$r_2 + r_3 = (a_1-4)n+7 \text{ and } r_2r_3 = 6. \tag{1.8}$$

In order to have distinct indices, (1.8b) implies that $(r_2, r_3) = (1,6), (2,3), (-2,-3)$. From (1.4), (1.5), and (1.8), one gets the following 3 cases for (a_1, a_2, a_3, a_4):

1. $$(a_1, a_2, a_3, a_4) = \left(4, 3-\frac{6}{n}, -12+\frac{12}{n}, 6-\frac{6}{n}\right), \tag{1.9}$$

2. $$(a_1, a_2, a_3, a_4) = \left(4-\frac{2}{n}, 3-\frac{2}{n}, -12+\frac{10}{n}, 6-\frac{6}{n}\right), \tag{1.10}$$

3. $$(a_1, a_2, a_3, a_4) = \left(4-\frac{12}{n}, 3+\frac{18}{n}, -12, 6-\frac{6}{n}\right). \tag{1.11}$$

In the case of double branch, let

$$\begin{aligned} &a_1+a_2+a_3+a_4-1=0,\\ &3a_1+2a_2+a_3-6\neq 0, \end{aligned} \tag{1.12}$$

and the roots of (1.4) be $\alpha_1 = n$ and $\alpha_2 = m$ such that $n \neq m$, and $n, m \in \mathbb{Z} - \{0\}$. Then (1.4) implies that

$$\begin{aligned} &3a_1+2a_2+a_3-6 = -\frac{6}{nm},\\ &11-2a_1-a_2 = 6\left(\frac{1}{n}+\frac{1}{m}\right). \end{aligned} \tag{1.13}$$

Similarly, substituting (1.6) into (1.3) gives the following equations for the Fuchs indices r_{ji} $(j = 1,2, i = 0,1,2,3)$:

$$\begin{aligned} &r_1(r_1+1)\left\{r_1^2 - [(a_1-4)n+7]r_1 + 6 - \frac{6n}{m}\right\} = 0,\\ &r_2(r_2+1)\left\{r_2^2 - [(a_1-4)m+7]r_2 + 6 - \frac{6m}{n}\right\} = 0, \end{aligned} \tag{1.14}$$

for $\alpha_1 = n$ and $\alpha_2 = m$, respectively. Therefore $r_{j0} = -1$ and $r_{j1} = 0 (j = 1,2)$. In order to have distinct indices, if $r_{j2}r_{j3} = p_j$, then p_j satisfy the following Diophantine equation:

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{6}. \tag{1.15}$$

Among the solutions of the Diophantine equation (1.15), $(p_1, p_2) = (3, -6), (4, -12), (5, -30), (8, 24)$ and $(12, 12)$ are the only ones leading to distinct Fuchs indices (resonances).

When $(p_1, p_2) = (3, -6)$, the distinct integer resonances for the first branch are $(r_{12}, r_{13}) = (1, 3)$. Then (1.14a) implies that

$$\begin{aligned} &r_{12} + r_{13} = 7 + (a_1+4)n = 4,\\ &r_{12}r_{13} = 6\left(1-\frac{n}{m}\right) = 3. \end{aligned} \tag{1.16}$$

The equations (1.16) give

$$a_1 = 4 - \frac{3}{n}, \quad m = 2n, \tag{1.17}$$

respectively. (1.14b) implies that, $r_{22} + r_{23} = 1$ and $r_{22}r_{23} = -6$, and hence the distinct integer resonances for the second branch are $(r_{22}, r_{23}) = (-2, 3)$. By using (1.12a) and (1.13), one obtains the following case:

4. $$(a_1, a_2, a_3, a_4) = \left(4-\frac{3}{n}, 3-\frac{3}{n}, -12+\frac{15}{n}-\frac{3}{n^2}, 6-\frac{9}{n}+\frac{3}{n^2}\right), \tag{1.18}$$

with the resonances $(r_{11}, r_{12}, r_{13}) = (0,1,3)$ and $(r_{21}, r_{22}, r_{23}) = (0,-2,3)$.

Similarly, when $(p_1, p_2) = (4, -12)$, the resonances for the first and second branches, respectively, are $(r_{11}, r_{12}, r_{13}) = (0,1,4)$ and $(r_{21}, r_{22}, r_{23}) = (0,-3,4)$ and we have the following case:

5. $$(a_1, a_2, a_3, a_4) = \left(4-\frac{2}{n}, 3-\frac{4}{n}, -12+\frac{14}{n}-\frac{2}{n^2}, 6-\frac{8}{n}+\frac{2}{n^2}\right). \tag{1.19}$$

When $(p_1, p_2) = (5, -30)$, by using the similar procedure, one obtains the following case:

6. $$(a_1, a_2, a_3, a_4) = \left(4-\frac{1}{n}, 3-\frac{5}{n}, -12+\frac{13}{n}-\frac{1}{n^2}, 6-\frac{7}{n}+\frac{1}{n^2}\right), \tag{1.20}$$

with the resonances $(r_{11}, r_{12}, r_{13}) = (0, 1, 5)$ and $(r_{21}, r_{22}, r_{23}) = (0, -5, 6)$.

When $(p_1, p_2) = (8, 24)$, distinct integer resonances for the first branch are $(r_{12}, r_{13}) = (1, 8)$, $(2, 4)$, and $(-2, -4)$, but only the resonances $(2, 4)$ lead to distinct integer resonances for the second branch. Hence, for this case the resonances are $(r_{11}, r_{12}, r_{13}) = (0, 2, 4)$, $(r_{21}, r_{22}, r_{23}) = (0, 4, 6)$, and

$$\textbf{7}.\quad (a_1,\, a_2,\, a_3,\, a_4) = \Big(4 - \frac{1}{n}\,,\, 3 - \frac{2}{n}\,,\, -12 + \frac{7}{n} + \frac{2}{n^2}\,,\, 6 - \frac{4}{n} - \frac{2}{n^2}\Big). \tag{1.21}$$

For $(p_1,\, p_2) = (12,\, 12)$, both branches have distinct integer resonances when the resonances for the first branch are $(r_{12}, r_{13}) = (3, 4)$. Then the coefficients $a_j (j = 1, \ldots, 4)$ and the resonances are as follows:

$$\textbf{8}.\quad (a_1,\, a_2,\, a_3,\, a_4) = \Big(4,\, 3,\, -12 + \frac{6}{n^2}\,,\, 6 - \frac{6}{n^2}\Big), \tag{1.22}$$

$(r_{j1}, r_{j2}, r_{j3}) = (0, 3, 4)$, $j = 1, 2$.

In the case of three branches, i. e. $a_1 + a_2 + a_3 + a_4 \neq 1$, let the roots of (1.4) be $\alpha_1 = n$, $\alpha_2 = m$ and $\alpha_3 = l$ such that $\alpha_i \neq \alpha_j$, $i \neq j$, and $n, m, l \in \mathbb{Z} - \{0\}$. Then (1.4) implies that

$$\begin{aligned} &a_1 + a_2 + a_3 + a_4 - 1 = \frac{6}{nml}, \\ &11 - 2a_1 - a_2 = 6\Big(\frac{1}{n} + \frac{1}{m} + \frac{l}{m}\Big), \\ &6 - 3a_1 - 2a_2 - a_3 = 6\Big(\frac{1}{nl} + \frac{1}{ml} + \frac{1}{nm}\Big). \end{aligned} \tag{1.23}$$

Substituting (1.6) into (1.3), we obtain the following equations for the Fuchs indices $r_{ji} (j = 1, 2, 3,\ i = 0, 1, 2, 3)$:

$$\begin{aligned} &r_1(r_1 + 1)\Big\{r_1^2 - [(a_1 - 4)n + 7]r_1 + 6\Big(1 - \frac{n}{m}\Big)\Big(1 - \frac{n}{l}\Big)\Big\} = 0, \\ &r_2(r_2 + 1)\Big\{r_2^2 - [(a_1 - 4)m + 7]r_2 + 6\Big(1 - \frac{m}{n}\Big)\Big(1 - \frac{m}{l}\Big)\Big\} = 0, \\ &r_3(r_3 + 1)\Big\{r_3^2 - [(a_1 - 4)l + 7]r_3 + 6\Big(1 - \frac{l}{m}\Big)\Big(1 - \frac{l}{n}\Big)\Big\} = 0, \end{aligned} \tag{1.24}$$

for $\alpha_1 = n$, $\alpha_2 = m$ and $\alpha_3 = l$, respectively. Therefore, the first two resonances for all three branches are $r_{j0} = -1$ and $r_{j1} = 0 (j = 1, 2, 3)$. If we let $p_j = r_{j2} r_{j3}$, then (1.24) imply that

$$\begin{aligned} p_1 &= 6\Big(1 - \frac{n}{m}\Big)\Big(1 - \frac{n}{l}\Big), \\ p_2 &= 6\Big(1 - \frac{m}{n}\Big)\Big(1 - \frac{m}{l}\Big), \\ p_3 &= 6\Big(1 - \frac{l}{m}\Big)\Big(1 - \frac{l}{n}\Big), \end{aligned} \tag{1.25}$$

and hence p_j satisfy the following Diophantine equation:

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{6}. \tag{1.26}$$

Equation (1.26) implies that at least one of p_j is a positive integer. Let $p_1 > 0$, since

$$p_1 p_2 p_3 = -6^3 \frac{(m - n)^2 (l - n)^2 (l - m)^2}{n^2 m^2 l^2}, \tag{1.27}$$

if we let $p_2 > 0$, then $p_3 < 0$. Among the solutions satisfying the conditions p_1, $p_2 > 0$, and $p_3 < 0$ of the Diophantine equation (1.26), $(p_1, p_2, p_3) = (6, N, -N)$ where $N \in \mathbb{Z}_+$ is the only one leading to distinct integer resonances for all three branches. If we let

$$\lambda = 6\frac{(m - n)(l - n)(l - m)}{nml}, \tag{1.28}$$

then equations (1.25) give

$$\begin{aligned} &\lambda n - p_1 m + p_1 l = 0, \\ &p_2 n + \lambda m - p_2 l = 0, \\ &-p_3 n + p_3 m + \lambda l = 0, \end{aligned} \tag{1.29}$$

and thus

$$\lambda^2 = -(p_1 p_2 + p_2 p_3 + p_1 p_3). \tag{1.30}$$

Therefore, when $(p_1, p_2, p_3) = (6, N, -N)$, $\lambda = \pm N$. For $\lambda = N$, (1.29) implies that $n = 0$ and $m = l$. Similarly for $\lambda = -N$,

$$m = \frac{(6 - N)n}{12}, \qquad l = \frac{(6 + N)n}{12}, \tag{1.31}$$

provided that $N \neq 6$, and $(6 \pm N)n/12$ are integers. Since $p_1 = 6$, then the possible integer resonances for the first branch are $(r_{12}, r_{13}) = (1,\, 6), (-2, -3)$

and $(2,3)$. When $(r_{12},r_{13})=(1,6)$, there are no integer resonances for the second and third branches. Therefore, we have the following two subcases: When $(r_{12},r_{13})=(-2,-3)$, the (1.24a) gives

$$r_{12}+r_{13}=(a_1-4)n+7=5. \tag{1.32}$$

Thus, $a_1=4-12/n$. Similarly, (1.24b) and (1.24c), respectively, imply

$$\begin{aligned} r_{22}+r_{23}&=1+N, & r_{22}r_{23}&=N,\\ r_{32}+r_{33}&=1-N, & r_{32}r_{33}&=-N. \end{aligned} \tag{1.33}$$

Therefore, the resonances for the second and third branches are $(r_{22},r_{23})=(1,N)$ and $(r_{32},r_{33})=(1,-N)$, providing that $N\neq 1$. The coefficients a_2,a_3, and a_4 can be determined from equations (1.23). Thus we have the following case:

$$\begin{aligned} \mathbf{9}.\quad (a_1,a_2,a_3,a_4)=\Bigg(&4-\frac{12}{n},3-\frac{216+18N^2}{(36-N^2)n},-12+\frac{1728}{(36-N^2)n}-\frac{1728}{(36-N^2)n^2},\\ &6+\frac{6N^2-1080}{(36-N^2)n}+\frac{1728}{(36-N^2)n^2}-\frac{846}{(36-N^2)n^3}\Bigg), \end{aligned} \tag{1.34}$$

such that $N\in\mathbb{Z}_+-\{1,6\}$, and n, $\frac{(6\pm N)n}{12}$ are non-zero integers. It should be noted that, as $N\to\infty$ this case reduce to the third case given by (1.11).

When $(r_{12},r_{13})=(2,3)$, by following the similar procedure we obtain the following case:

$$\begin{aligned} \mathbf{10}.\quad (a_1,a_2,a_3,a_4)=\Bigg(&4-\frac{2}{n},3+\frac{2k^2-26}{(1-k^2)n},-12+\frac{58-10k^2}{(1-k^2)n}-\frac{48}{(1-k^2)n^2},\\ &6+\frac{6k^2-30}{(1-k^2)n}+\frac{48}{(1-k^2)n^2}-\frac{24}{(1-k^2)n^3}\Bigg), \end{aligned} \tag{1.35}$$

where $k=N/6$, $k\in\mathbb{Z}_+-\{1,6\}$ providing that n, $\frac{(1\pm k)n}{2}$ are non-zero integers. It should be noted that, as $k\to\infty$, this case reduces to the second case given by (1.10).

Moreover, as $n\to\pm\infty$, we have the following case:

$$\mathbf{11}.\quad (a_1,\,a_2,\,a_3,\,a_4)=(4,\,3,\,-12,\,6). \tag{1.36}$$

Thus, we have eleven cases, (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36), and all the corresponding equations pass the Painlevé test. Moreover, if one lets $u=\dot{y}/y$, (1.3) yields the following third order polynomial type equations:

$$\begin{aligned} \dddot{u}&=(a_1-4)u\ddot{u}+(a_2-3)\dot{u}^2\\ &\quad+(3a_1+2a_2+a_3-6)u^2\dot{u}\\ &\quad+(a_1+a_2+a_3+a_4-1)u^4. \end{aligned} \tag{1.37}$$

For case 11, (1.37) yields $\dddot{u}=0$, and for the cases 1–10 (1.37) yields an equation of Painlevé type with leading order $\alpha=-1$, and $u_0=n$, i.e. $u\cong n(t-t_0)^{-1}$ as $t\to t_0$. The equations in u obtained form (1.37) for the cases 1 – 10 were examined in [4, 6, 11, 13].

In the following sections, the "simplified equations" that retain only the leading terms as $z\to z_0$ will be considered for $\alpha=-4,-3,-2$ and -1 with positive distinct resonances, and some simplified equations for $\alpha=-4,-3,-2$ with negative resonances.

2. Leading Order $\alpha=-2$

(1.2) contains the leading terms for any $\alpha=n\in\mathbb{Z}-\{0\}$ as $z\to z_0$, if we do not take into account F. In this section, we consider the case $\alpha=-2$. By adding the terms of order -6 or greater as $z\to z_0$, we obtain the equation

$$\begin{aligned} y^{(4)}&=a_1\frac{y'y'''}{y}+a_2\frac{(y'')^2}{y}+a_3\frac{(y')^2y''}{y^2}\\ &\quad+a_4\frac{(y')^4}{y^3}+b_1yy''+b_2(y')^2\\ &\quad+b_3y^3+F_1(y,y',y'',y''',z), \end{aligned} \tag{2.1}$$

where $b_i (i=1,2,3)$ are constants and F_1 contains the terms of order -5 or greater as $z\to z_0$. We consider the case of $F_1=0$, i.e., the simplified equation for $\alpha=-2$.

Suppose that (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36) hold, and substitute $y \cong y_0(z-z_0)^{-2} + \beta(z-z_0)^{r-2}$ into (2.1) with $F_1 = 0$. Then we obtain the following equations for the Fuchs indices r and y_0 [22]:

$$\begin{aligned} &Q(r) = (r+1)\{r^3 + (2a_1-15)r^2 \\ &-(20a_1+12a_2+4a_3+b_1y_0-86)r \\ &+2[48a_1+36a_2+24a_3+16a_4 \\ &+(3b_1+2b_2)y_0-120]\} = 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &b_3y_0^2 + 2(3b_1+2b_2)y_0 \\ &+4(12a_1+9a_2+6a_3+4a_4-30) = 0, \end{aligned} \tag{2.3}$$

(2.3) implies, that, in general, there are two branches if $b_3 \neq 0$. Now, we determine $y_{0j}(j=1,2)$ and b_i for each case of (a_1,a_2,a_3,a_4) such that one branch is the principal branch, i.e., all the resonances are positive distinct integers (except $r_0 = -1$). If $r_0 = -1$ and $r_{ji}(i=1,2,3)$ then (2.2) implies that

$$\begin{aligned} &\sum_{i=1}^{3} r_{ji} = -(2a_1-15), \\ &\sum_{\substack{i,k=1\\ i\neq k}}^{3} r_{ji}r_{jk} = -(20a_1+12a_2+4a_3 \\ &\qquad\qquad + b_1y_{0j}-86), \\ &\prod_{i=1}^{3} r_{ji} = -2[48a_1+36a_2+24a_3+16a_4 \\ &\qquad\qquad + (3b_1+2b_2)y_{0j}-120],\ j=1,2 \end{aligned} \tag{2.4}$$

providing that the right hand sides of (2.4) are positive integers for at least one of y_{0j}. According to the number of branches, the following cases should be considered separately:

Case I. $b_3 = 0$: In this case, there is one branch. $(a_1,a_2,a_3,a_4) = (4,\ 3-\frac{6}{n},\ -12+\frac{12}{n},\ 6-\frac{6}{n})$, (2.4) gives

$$\begin{aligned} &\sum_{i=1}^{3} r_i = 7,\ \sum_{\substack{i,j=1\\ i\neq j}}^{3} r_ir_j = 18+\frac{24}{n}-b_1y_0, \\ &\prod_{i=1}^{3} r_i = 12+\frac{24}{n}. \end{aligned} \tag{2.5}$$

In order to have a principal branch, n takes the values of $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$. Among these values of n, only for $n=-6,-2$ there exists a principal branch with the resonances $(r_1,r_2,r_3) = (1,2,4)$ and $(0,1,6)$, respectively. Then, b_1 and b_2 can be determined from equations (2.5b) and (2.3), respectively. For $n=-2$, $(b_1,b_2,b_3) = (0,0,0)$ that is, no additional leading term. For $n=-6$, we have the following simplified equation:

$$\begin{aligned} y^{(4)} = 4\frac{y'y'''}{y} + 4\frac{(y'')^2}{y} - 14\frac{(y')^2y''}{y^2} \\ +\ 7\frac{(y')^4}{y^3} + 2(y')^2. \end{aligned} \tag{2.6}$$

(2.6) does not pass the Painlevé test since the compatibility condition at $r_2 = 2$ is not satisfied identically. When $\alpha = -2$, in the case of the single branch for all cases (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35) and (1.36) there are no additional leading terms and the simplified equations are the same as the reduced equations (1.3) for $n=-2$.

Case II. $b_3 \neq 0$: If y_{0j} $(j=1,2;\ y_{01} \neq y_{02})$, are the roots of (2.3), and (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to y_{0j}, then let

$$\prod_{i=1}^{3} r_{ji} = P(y_{0j}) = p_j,\ j=1,2, \tag{2.7}$$

where

$$\begin{aligned} P(y_{0j}) = 2[120-48a_1-36a_2-24a_3 \\ -16a_4-(3b_1+2b_2)y_{0j}],\ j=1,2 \end{aligned} \tag{2.8}$$

and $p_j \in \mathbb{Z} - \{0\}$. In order to have a principal branch, at least one of the p_j should be a positive integer. (2.3) gives

$$b_3 = -\frac{2q}{y_{01}y_{02}},\ 3b_1+2b_2 = q\left(\frac{1}{y_{01}}+\frac{1}{y_{02}}\right), \tag{2.9}$$

where $q = 60-24a_1-18a_2-12a_3-8a_4$. Then (2.8) can be written as

$$p_j = 2q\left(1-\frac{y_{0j}}{y_{0k}}\right),\ j,k=1,2,\ j\neq k. \tag{2.10}$$

If $p_1p_2 \neq 0$ and $q \neq 0$, then p_j satisfy the following simple hyperbolic type of Diophantine equation:

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{2q}. \tag{2.11}$$

The general solution of (2.11) is given as

$$p_1 = 2q-d_i, \quad p_2 = 2q\left(1-\frac{2q}{d_i}\right), \tag{2.12}$$

where $\{d_i\}$ is the set of divisors of $4q^2$. For each case (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35) and (1.36), from (2.4a) one can find the possible resonances of the principal branch, then p_2 can be obtained from the Diophantine equation (2.11). Once p_2 is known, possible resonances for the second branch satisfying the conditions (2.10) and (2.4c) can be determined. Then the coefficients b_1 and b_2, b_3 of the additional leading terms can be determined by using (2.4c) and (2.9), respectively.

For the double branch case, we have only the following equations which have the additional leading terms:

For case 4, (1.18):

$$\begin{aligned} &y^{(4)} = \frac{y'y'''}{y} + 7yy'' - 3y^3, \\ &y_{01} = 2,\ y_{02} = 12,\ (r_{11}, r_{12}, r_{13}) = (2, 5, 6), \\ &(r_{21}, r_{22}, r_{23}) = (-5, 6, 12), \end{aligned} \tag{2.13}$$

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} + 2\frac{(y'')^2}{y} - \frac{22}{3}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{10}{3}\frac{(y')^4}{y^3} + 8yy'' - (y')^2 - 6y^3, \\ y_{01} &= 2/3,\ y_{02} = 20/3, \\ (r_{11}, r_{12}, r_{13}) &= (2, 3, 4), \\ (r_{21}, r_{22}, r_{23}) &= (-5, 6, 8). \end{aligned} \tag{2.14}$$

For case 5, (1.19):

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} + \frac{(y'')^2}{y} - \frac{11}{2}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{5}{2}\frac{(y')^4}{y^3} + 5yy'' - \frac{5}{2}(y')^2 - 2y^3, \\ y_{01} &= 2,\ y_{02} = 8, \\ (r_{11}, r_{12}, r_{13}) &= (2, 3, 4), \\ (r_{21}, r_{22}, r_{23}) &= (-3, 4, 8). \end{aligned} \tag{2.15}$$

For case 6, (1.20):

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} - 2\frac{(y'')^2}{y} + 24yy'' \\ &\quad - 18(y')^2 - 24y^3, \\ y_{01} &= 1,\ y_{02} = 2, \\ (r_{11}, r_{12}, r_{13}) &= (2, 3, 4), \\ (r_{21}, r_{22}, r_{23}) &= (-2, 3, 8). \end{aligned} \tag{2.16}$$

For case 8, (1.22);

$$\begin{aligned} y^{(4)} &= 4\frac{y'y'''}{y} + 3\frac{(y'')^2}{y} - \frac{93}{8}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{45}{8}\frac{(y')^4}{y^3} + 5yy'' - \frac{5}{2}(y')^2 - 4y^3, \\ y_{01} &= 1/2,\ y_{02} = 9/2, \\ (r_{11}, r_{12}, r_{13}) &= (1, 2, 4), \\ (r_{21}, r_{22}, r_{23}) &= (-3, 4, 6). \end{aligned} \tag{2.17}$$

For case 10, (1.35):

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + \frac{3}{2}\frac{(y'')^2}{y} \\ &\quad - 2\frac{(y')^2y''}{y^2} + 18y^3, \\ y_{0j}^2 &= 1,\ (r_{j1}, r_{j2}, r_{j3}) = (2, 3, 6), \\ j &= 1,\ 2, \end{aligned} \tag{2.18}$$

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} + \frac{5}{2}\frac{(y'')^2}{y} - \frac{15}{2}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{25}{8}\frac{(y')^4}{y^3} + 6yy'' - 4\left[(y')^2 + y^3\right], \\ y_{01} &= 1,\ y_{02} = 4, \\ (r_{11}, r_{12}, r_{13}) &= (1, 2, 6), \\ (r_{21}, r_{22}, r_{23}) &= (-2, 3, 8), \end{aligned} \tag{2.19}$$

$$\begin{aligned} y^{(4)} &= \frac{9}{2}\frac{y'y'''}{y} + \left(\frac{7}{2} + \frac{6}{1-k^2}\right)\frac{(y'')^2}{y} \\ &\quad - \left(\frac{29}{2} + \frac{15}{1-k^2}\right)\frac{(y')^2y''}{y^2} \\ &\quad + \left(\frac{15}{2} + \frac{75}{1-k^2}\right)\frac{(y')^4}{y^3} \\ &\quad + (k^2 + 11)yy'' \\ &\quad - 15(y')^2 + 6(1-k^2)y^3, \\ y_{01} &= -1/(1-k^2),\ y_{02} = -k^2/(1-k^2), \\ (r_{11}, r_{12}, r_{13}) &= (1, 2, 3), \\ (r_{21}, r_{22}, r_{23}) &= (6, k, -k),\ k \neq 1, 6. \end{aligned} \tag{2.20}$$

For all the other cases, the simplified equations are the same as the reduced equations (1.3) with the coefficients for $n = -2$.

3. Leading Order $\alpha = -1$

$\alpha = -1$ is also a possible leading order of (1.2) if $F = 0$. By adding terms of order -5, we obtain the simplified equation with the leading order $\alpha = -1$

$$\begin{aligned} y^{(4)} &= a_1\frac{y'y'''}{y} + a_2\frac{(y'')^2}{y} + a_3\frac{(y')^2y''}{y^2} \\ &\quad + a_4\frac{(y')^4}{y^3} + c_1yy''' + c_2y'y'' + c_3\frac{(y')^3}{y} \\ &\quad + c_4y^2y'' + c_5y(y')^2 + c_6y^3y' + c_7y^5, \end{aligned} \tag{3.1}$$

where c_k $(k = 1, \ldots, 7)$ are constants.

Suppose that (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36) hold. If we substitute $y \cong y_0(z-z_0)^{-1} + \beta(z-z_0)^{r-1}$ into (3.1), then we obtain the following equations for the Fuchs indices r and y_0:

$$\begin{aligned} Q(r) = (r+1)\{r^3 &+ (a_1 - c_1 y_0 - 11)r^2 \\ &- [7a_1 + 4a_2 + a_3 - (7c_1 + c_2)y_0 + c_4 y_0^2 - 46]r \\ &+ 24a_1 + 16a_2 + 8a_3 + 4a_4 - 96 \\ &- 3(6c_1 + 2c_2 + c_3)y_0 \\ &+ 2(2c_4 + c_5)y_0^2 - c_6 y_0^3\} = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &c_7 y_0^4 - c_6 y_0^3 + (c_5 + 2c_4)y_0^2 \\ &-(c_3 + 2d_2 + 6c_1)y_0 \\ &+6a_1 + 4a_2 + 2a_3 + a_4 - 24 = 0. \end{aligned} \tag{3.3}$$

(3.3) implies, that, in general, there are four branches if $c_7 \neq 0$. Now, we determine $y_{0j}(j = 1, \ldots, 4)$, and c_k for each case of (a_1, a_2, a_3, a_4) such that one branch is the principal branch. If $r_0 = -1$ and r_{ji} $(j = 1, \ldots, 4,\ i = 1, 2, 3)$ then (3.2) implies that

$$\begin{aligned} &\sum_{i=1}^{3} r_{ji} = 11 - a_1 + c_1 y_0, \\ &\sum_{\substack{i,k=1 \\ i\neq k}}^{3} r_{ji} r_{jk} = 46 - 7a_1 - 4a_2 - a_3 \\ &\qquad\qquad + (7c_1 + c_2)y_0 - c_4 y_0^2, \\ &\prod_{i=1}^{3} r_{ji} = 96 - 24a_1 - 16a_2 - 8a_3 - 4a_4 \\ &\qquad\qquad + 3(6c_1 + 2c_2 + c_3)y_0 \\ &\qquad\qquad - 2(2c_4 + c_5)y_0^2 + c_6 y_0^3, \\ &j = 1, \ldots, 4. \end{aligned} \tag{3.4}$$

According to number of branches, the following cases should be considered separately:

Case I. $c_4 = c_5 = c_6 = c_7 = 0$: In this case, there is one branch. From (3.4c) one can find the possible resonances of the single principal branch for each case (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36). Then the coefficients c_1, c_2, and c_3 can be determined by using the equations (3.3), (3.4a) and (3.4b). For the cases (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), and (1.22), and for all possible values of n, there is no simplified equation which passes the Painlevé test. Hence, for these cases the simplified equations are the same as the reduced equations (1.3) with the coefficients for $n = -1$. For the cases (1.34) and (1.35), the right hand side of (3.4c) depends on N and k, respectively. When (1.34) holds for $n = 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, and when (1.35) holds for $n = 1, \pm 2$, the simplified equations do not pass the Painlevé test. Hence, for these cases and for these particular values of n, the simplified equations are the same as the reduced equation (1.3) with the coefficients for $n = -1$. For the case (1.36), we obtain the following simplified equation:

$$\begin{aligned} y^{(4)} = 4\frac{y'y'''}{y} &+ 3\frac{(y'')^2}{y} - 12\frac{(y')^2 y''}{y^2} \\ &+ 6\frac{(y')^4}{y^3} - yy''', \end{aligned} \tag{3.5}$$

$$y_{01} = 1,\ (r_1, r_2, r_3) = (1, 2, 3).$$

Case II. $c_6 = c_7 = 0$: In this case, there are two branches. If $y_{0j}(j = 1, 2;\ y_{01} \neq y_{02})$ are the roots of (3.3), and $(r_{j1}(r_{j2}, r_{j3})$ are the resonances corresponding to y_{0j}, then let

$$\prod_{i=1}^{3} r_{ji} = P(y_{0j}) = p_j,\ j = 1, 2, \tag{3.6}$$

where

$$\begin{aligned} P(y_{0j}) = -[&24a_1 + 16a_2 + 8a_3 + 4a_4 - 96 \\ &- 3(6c_1 + 2c_2 + c_3)y_{0j} \\ &+ 2(2c_4 + c_5)y_{0j}^2],\ j = 1, 2, \end{aligned} \tag{3.7}$$

and $p_j \in \mathbb{Z} - \{0\}$. In order to have a principal branch, at least one of the p_j should be a positive integer. (3.3) gives

$$\begin{aligned} &c_5 + 2c_4 = -\frac{s}{y_{01}y_{02}}, \\ &c_3 + 2c_2 + c_1 = -s\Big(\frac{1}{y_{01}} + \frac{1}{y_{02}}\Big), \end{aligned} \tag{3.8}$$

where

$$s = 24 - 6a_1 - 4a_2 - 2a_3 - a_4. \tag{3.9}$$

Then (3.7) can be written as

$$p_j = s\Big(1 - \frac{y_{0j}}{y_{0k}}\Big),\ j, k = 1, 2,\ j \neq k. \tag{3.10}$$

If $\prod p_j \neq 0$ and $s \neq 0$, then p_j satisfy the following simple hyperbolic type of Diophantine equation

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{s}\ . \tag{3.11}$$

The general solution of (2.11) is given as

$$p_1 = s - d_i,\ p_2 = s\Big(1 - \frac{s}{d_i}\Big), \tag{3.12}$$

where $\{d_i\}$ is the set of divisors of s^2. For each case (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36), from (3.4a) one can find the possible resonances of the principal branch, consequently p_1, then p_2 can be obtained from the Diophantine equation (3.11). Once p_2 is known, possible resonances for the second branch satisfying the conditions (3.4) can be determined. Then the coefficients c_k of the additional leading terms can be determined by using (3.4) and (3.8). One should consider the cases $c_1 = 0$ and $c_1 \neq 0$ separately.

IIa. $c_1 = 0$: For this case, we have the following equations which admit additional leading terms:

For case 2, (1.10):

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + \frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} \\ &\quad + 4\left[y^2y'' + (y')^2\right], \\ y_{0j}^2 &= 1,\ (r_{j1}, r_{j2}, r_{j3}) = (2,3,4),\ j = 1,2. \end{aligned} \tag{3.13}$$

For case 3, (1.11):

$$\begin{aligned} y^{(4)} &= 5\frac{y'y'''}{y} + \frac{3}{2}\frac{(y'')^2}{y} - 12\frac{(y')^2y''}{y^2} \\ &\quad + \frac{13}{2}\frac{(y')^4}{y^3} - 5\Big[y'y'' - \frac{(y')^3}{y}\Big] \\ &\quad + y^2y'' - \frac{15}{2}y(y')^2, \\ y_{01} &= 1,\ y_{02} = -11, \\ (r_{11}, r_{12}, r_{13}) &= (1,2,3), \\ (r_{21}, r_{22}, r_{23}) &= (-2,-3,11). \end{aligned} \tag{3.14}$$

For all the other cases, the simplified equations are the same as the reduced equations (1.3) with the coefficients for $n = -1$.

IIb. $c_1 \neq 0$: When $n = 1$, we obtain the following equations which admit the additional leading terms:

For case 1, (1.9):

$$\begin{aligned} y^{(4)} &= 4\frac{y'y'''}{y} - 3\frac{(y'')^2}{y} - yy''' \\ &\quad - 6\Big[2y'y'' - 2\frac{(y')^3}{y} + y(y')^2\Big], \\ y_{01} &= 1,\ y_{02} = 2, \\ (r_{11}, r_{12}, r_{13}) &= (1,2,3), \\ (r_{21}, r_{22}, r_{23}) &= (-2,1,6). \end{aligned} \tag{3.15}$$

For case 2, (1.10):

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + \frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} \\ &\quad - 3yy''' - 2\left[y^2y'' + y(y')^2\right], \\ y_{01} &= 1,\ y_{02} = 2, \\ (r_{11}, r_{12}, r_{13}) &= (1,2,3), \\ (r_{21}, r_{22}, r_{23}) &= (-2,2,3). \end{aligned} \tag{3.16}$$

For all the other cases, the simplified equations are the same as the reduced equations (1.3) with the coefficients for $n = -1$.

Case III. $c_7 = 0$: In this case, there are three branches. If y_{0j} ($j = 1,2,3$; $y_{0j} \neq y_{0k}$, $j \neq k$), are the roots of (3.3), and (r_{j1}, r_{j2}, r_{j3}) are the resonances corresponding to y_{0j}, then let

$$\prod_{i=1}^{3} r_{ji} = P(y_{0j}) = p_j,\ j = 1,2,3, \tag{3.17}$$

where

$$\begin{aligned} P(y_{0j}) = -[&24a_1 + 16a_2 + 8a_3 + 4a_4 \\ &- 96 - 3(6c_1 + 2c_2 + c_3)y_{0j} \\ &+ 2(2c_4 + c_5)y_{0j}^2 - c_6y_{0j}^3], \\ j = 1,2,3,& \end{aligned} \tag{3.18}$$

and $p_j \in \mathbb{Z} - \{0\}$. In order to have a principal branch, at least one of the p_j should be a positive integer. (3.3) gives

$$c_6 = -s\prod_{j=1}^{3}\frac{1}{y_{0j}},\ c_5 + 2c_4 = -s\prod_{j=1}^{3}\frac{1}{y_{0j}}\Big(\sum_{j=1}^{3} y_{0j}\Big),$$

$$c_3 + 2c_2 + 6c_1 = -s\prod_{j=1}^{3}\frac{1}{y_{0j}}\Big(\sum_{\substack{j,k=1,\\ j\neq k}}^{3} y_{0j}y_{0k}\Big), \tag{3.19}$$

where s is given in (3.9). Then (3.18) can be written as

$$p_j = s \prod_{\substack{k=1\\ k\neq j}}^{3} \left(1 - \frac{y_{0j}}{y_{0k}}\right), \ j = 1, \ldots, 4. \tag{3.20}$$

If $\prod p_j \neq 0$ and $s \neq 0$, then p_j satisfy the following Diophantine equation:

$$\sum_{j=1}^{3} \frac{1}{p_j} = \frac{1}{s} \,. \tag{3.21}$$

One should consider the cases $c_1 = 0$ and $c_1 \neq 0$ separately.

IIIa. $c_1 = 0$: For each case (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35), and (1.36), from (3.4a) one can find the possible resonances of the principal branch, consequently p_1, then the Diophantine equation (3.21) can be reduced to the following simple hyperbolic type which can be solved in closed form:

$$\frac{1}{p_2} + \frac{1}{p_3} = \frac{k_1}{k_2}, \tag{3.22}$$

where $k_1 = p_1 - s$ and $k_2 = sp_1$. The general solution of (3.22) is given as

$$p_2 = \frac{k_2 + d_i}{k_1}, \ p_3 = \frac{k_2(k_2 + d_i)}{k_1 d_i}, \tag{3.23}$$

where $\{d_i\}$ is the set of divisors of k_2^2. For the cases (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), and (1.36), and for all $n \neq -1$ there is no simplified equation which passes the Painlevé test. For the cases (1.9), (1.22), (1.34) and (1.35) when $n = 1, \pm 2, \pm 3$, $n^2 = 4, 9$, $n = 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, $N = 2, 3$ and $n = 1, 2, -2$, $k = 2, 3, 4$, respectively, there is no simplified equation which passes the Painlevé test.

IIIb. $c_1 \neq 0$: Equation (3.20) implies that

$$\prod_{j=1}^{3} p_j = -s^3 \frac{(y_{01} - y_{02})^2 (y_{02} - y_{03})^2 (y_{03} - y_{01})^2}{y_{01}^2 y_{02}^2 y_{03}^2}. \tag{3.24}$$

Hence, if $s > 0$, let $p_1 > 0$ then $p_2 > 0$, $p_3 < 0$, and if $s < 0$, let $p_1 > 0$ then $p_2, p_3 < 0$. For $s > 0$, suppose that $p_1 < p_2$, then from (3.21) one obtains $p_1 < 2s$. Similarly, for $s < 0$, one obtains $p_1 < s$. Since for both cases $s > 0$ and $s < 0$, $p_1 > 0$ and bounded by $2s$ and s from above, respectively, the Diophantine equation (3.21) can be reduced to a simple hyperbolic type in p_2 and p_3 which has the closed form solution given by (3.23). Once p_2 and p_3 are known, the possible resonances for the second and third branches satisfying the conditions (3.4) can be determined. Then the coefficients c_k of the additional leading terms can be determined by using (3.4) and (3.19).

For $n = 2$ (for case 9, (1.34): $n = 2$, $N = 18, 24$, and for case 10, (1.35): $n = 2$, $k = 3, 4$), we have the following simplified equations which admit the additional leading terms:

For case 4, (1.18):

$$\begin{aligned} y^{(4)} &= \frac{5}{2}\frac{y'y'''}{y} + \frac{3}{2}\frac{(y'')^2}{y} - \frac{21}{4}\frac{(y')^2 y''}{y^2} \\ &\quad + \frac{9}{4}\frac{(y')^4}{y^3} + yy''' + \frac{3}{5}y'y'' - \frac{3}{10}\frac{(y')^3}{y} \\ &\quad - \frac{9}{4}y\left[yy'' + (y')^2\right] + \frac{6}{125}y^3 y', \end{aligned} \tag{3.25}$$

$y_{01} = -5/2, \ y_{02} = -25/2, \ y_{03} = -15/2,$
$(r_{11}, r_{12}, r_{13}) = (1, 2, 3),$
$(r_{21}, r_{22}, r_{23}) = (-5, -2, 3),$
$(r_{31}, r_{32}, r_{33}) = (-3, 1, 3).$

For case 5, (1.19):

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} + \frac{(y'')^2}{y} - \frac{11}{2}\frac{(y')^2 y''}{y^2} \\ &\quad + \frac{5}{2}\frac{(y')^4}{y^3} + yy''' + \frac{3}{2}y'y'' - \frac{5}{4}\frac{(y')^3}{y} \\ &\quad - \frac{1}{8}y\left[3yy'' + 5(y')^2\right] + \frac{1}{16}y^3 y', \end{aligned} \tag{3.26}$$

$y_{01} = -2, \ y_{02} = -14, \ y_{03} = -6,$
$(r_{11}, r_{12}, r_{13}) = (1, 2, 3),$
$(r_{21}, r_{22}, r_{23}) = (-7, -3, 4),$
$(r_{31}, r_{32}, r_{33}) = (-3, 1, 4).$

For case 6, (1.20):

$$\begin{aligned} y^{(4)} &= \frac{7}{2}\frac{y'y'''}{y} + \frac{1}{2}\frac{(y'')^2}{y} - \frac{23}{4}\frac{(y')^2 y''}{y^2} \\ &\quad + \frac{11}{4}\frac{(y')^4}{y^3} + yy''' + 3y'y'' - \frac{17}{6}\frac{(y')^3}{y} \\ &\quad - \frac{1}{3}y\Big[yy'' + \frac{11}{3}(y')^2 - \frac{2}{9}y^2 y'\Big], \end{aligned}$$

$$\begin{aligned}&y_{01}=-3/2,\ y_{02}=-39/2,\ y_{03}=-9/2,\\&(r_{11},r_{12},r_{13})=(1,2,3),\\&(r_{21},r_{22},r_{23})=(-13,-5,6),\\&(r_{31},r_{32},r_{33})=(-3,1,5).\end{aligned}\tag{3.27}$$

For case 7, (1.21):

$$\begin{aligned}y^{(4)}=&\frac{7}{2}\frac{y'y'''}{y}+2\frac{(y'')^2}{y}-8\frac{(y')^2y''}{y^2}+\frac{7}{2}\frac{(y')^4}{y^3}\\&+yy'''-\frac{1}{3}\frac{(y')^3}{y}-\frac{2}{9}y\Big[(y')^2+\frac{2}{3}y^2y'\Big],\\&y_{01}=-3/2,\ y_{02}=15/2,\ y_{03}=-9/2,\\&(r_{11},r_{12},r_{13})=(1,2,3),\\&(r_{21},r_{22},r_{23})=(4,5,6),\\&(r_{31},r_{32},r_{33})=(-3,2,4).\end{aligned}\tag{3.28}$$

For case 8, (1.22):

$$\begin{aligned}y^{(4)}=&4\frac{y'y'''}{y}+3\frac{(y'')^2}{y}-\frac{21}{2}\frac{(y')^2y''}{y^2}\\&+\frac{9}{2}\frac{(y')^4}{y^3}+yy'''-3y'y''+\frac{3}{2}\frac{(y')^3}{y}\\&+\frac{3}{2}y\left[yy''+(y')^2\right]-\frac{3}{2}y^3y',\\&y_{01}=-1,\ y_{02}=1,\ y_{03}=-3,\\&(r_{11},r_{12},r_{13})=(1,2,3),\\&(r_{21},r_{22},r_{23})=(1,3,4),\\&(r_{31},r_{32},r_{33})=(-3,3,4).\end{aligned}\tag{3.29}$$

Case IV. $c_7\neq 0$: In this case, there are four branches. If y_{0j} $(j=1,\dots,4;\ y_{0j}\neq y_{0k},\ j\neq k)$ are the roots of (3.3), and (r_{j1},r_{j2},r_{j3}) are the resonances corresponding to y_{0j}, then let

$$\prod_{i=1}^{3}r_{ji}=P(y_{0j})=p_j,\ j=1,\dots,4,\tag{3.30}$$

where

$$\begin{aligned}&P(y_{0j})=-[24a_1+16a_2+8a_3+4a_4\\&-96-3(6c_1+2c_2+c_3)y_{0j}\\&+2(2c_4+c_5)y_{0j}^2-c_6y_{0j}^3],\ j=1,\dots,4,\end{aligned}\tag{3.31}$$

and $p_j\in\mathbb{Z}-\{0\}$. In order to have a principal branch, at least one of the p_j should be a positive integer. When $c_1=0$, (3.3) gives

$$\begin{aligned}&c_7=-s\prod_{j=1}^{4}\frac{1}{y_{0j}},\ c_6=-s\prod_{j=1}^{4}\frac{1}{y_{0j}}\bigg(\sum_{j=1}^{4}y_{0j}\bigg),\\&c_5+2c_4=-s\prod_{j=1}^{4}\frac{1}{y_{0j}}\bigg(\sum_{\substack{j,k=1,\\ j\neq k}}^{4}y_{0j}y_{0k}\bigg),\\&c_3+2c_2=-s\prod_{j=1}^{4}\frac{1}{y_{0j}}\bigg(\sum_{\substack{j,k,l=1,\\ j\neq k\neq l}}^{4}y_{0j}y_{0k}y_{0l}\bigg),\end{aligned}\tag{3.32}$$

where s is given in (3.9). Then (3.31) can be written as

$$p_j=s\prod_{\substack{k=1\\k\neq j}}^{4}\bigg(1-\frac{y_{0j}}{y_{0k}}\bigg),\qquad j=1,\dots,4.\tag{3.33}$$

If $\prod p_j\neq 0$ and $s\neq 0$, then p_j satisfy the following Diophantine equation:

$$\sum_{j=1}^{4}\frac{1}{p_j}=\frac{1}{s}.\tag{3.34}$$

If we let $p_1=p_2$ and $p_3=p_4$, then (3.34) can be reduced to a simple hyperbolic type of the Diophantine equation which admits the closed form solution (3.12). When $p_1=p_2$ and $p_3=p_4$, such that $p_1>0$, for each case (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35) and (1.36), from (3.4a) one can find the possible resonances for the branches satisfying the conditions (3.4). Then the coefficients c_k $(k=2,\dots,7)$ of the additional leading terms can be determined by using (3.4) and (3.32).

For this particular case, we have the following equations which admit additional leading terms:

For case 5, (1.19):

$$\begin{aligned}&y^{(4)}=2\frac{y'y'''}{y}-\frac{(y'')^2}{y}+5y^2y''-y^5,\ y_{01}^2=2,\\&y_{02}=-y_{01},\ y_{03}^2=8,\ y_{04}=-y_{03},\\&(r_{j1},r_{j2},r_{j3})=(2,3,4),\ j=1,2,\\&(r_{j1},r_{j2},r_{j3})=(-3,4,8),\ j=3,4.\end{aligned}\tag{3.35}$$

For case 7, (1.21):

$$y^{(4)}=3\frac{y'y'''}{y}+\frac{(y'')^2}{y}-3\frac{(y')^2y''}{y^2}+5y^2y''-2y^5,$$

$$y_{01}^2 = 1,\ y_{02} = -y_{01},\ y_{03}^2 = 4,\ y_{04} = -y_{03},$$
$$(r_{j1}, r_{j2}, r_{j3}) = (1, 3, 4),\ j = 1, 2, \tag{3.36}$$
$$(r_{j1}, r_{j2}, r_{j3}) = (-2, 4, 6),\ j = 3, 4.$$

For case 8, (1.22):

$$\begin{aligned} y^{(4)} &= 4\frac{y'y'''}{y} + 3\frac{(y'')^2}{y} - \frac{21}{2}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{9}{2}\frac{(y')^4}{y^3} + \frac{5}{2}y^2y'' - \frac{1}{2}y^5, \\ y_{01}^2 &= 1,\ y_{02} = -y_{01},\ y_{03}^2 = 9,\ y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 2, 4),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (-3, 4, 6),\ j = 3, 4. \end{aligned} \tag{3.37}$$

If one lets $y = 1/u$, then (3.37) gives

$$\begin{aligned} u^{(4)} &= 4\frac{u'u'''}{u} + 3\frac{(u'')^2}{u} - \frac{21}{2}\frac{(u')^2u''}{u^2} \\ &\quad + \frac{9}{2}\frac{(u')^4}{u^3} + 5\frac{u''}{u^3} - 10\frac{(u')^2}{u^3} + \frac{2}{u^3}. \end{aligned} \tag{3.38}$$

The canonical form (equation also contains the terms of order -4 or greater as $z \to z_0$) of (3.38) was also given in [18, 20].

For case 10, (1.35):

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + 2\frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} \\ &\quad + 6y^2y'' - 2y(y')^2 - 2y^5, \\ y_{01}^2 &= 1, y_{02} = -y_{01}, y_{03}^2 = 4, y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 2, 6),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (-2, 3, 8),\ j = 3, 4. \end{aligned} \tag{3.39}$$

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + 2\frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} \\ &\quad + 3y^2y'' + \frac{5}{2}y(y')^2 - \frac{1}{2}y^5, \\ y_{01}^2 &= 1, y_{02} = -y_{01}, y_{03}^2 = 16, y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 3, 5),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (-5, 6, 8),\ j = 3, 4. \end{aligned} \tag{3.40}$$

$$\begin{aligned} y^{(4)} &= 2\frac{y'y'''}{y} + \frac{3}{2}\frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} \\ &\quad + 5y^2y'' + \frac{5}{2}y(y')^2 - \frac{5}{2}y^5, \\ y_{01}^2 &= 1, y_{02} = -y_{01}, y_{03}^2 = 4, y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 3, 5),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (-2, 5, 6),\ j = 3, 4. \end{aligned} \tag{3.41}$$

The canonical form of (3.41) was given in [18 – 21].

$$\begin{aligned} y^{(4)} &= 3\frac{y'y'''}{y} + \frac{7}{2}\frac{(y'')^2}{y} - \frac{17}{2}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{27}{8}\frac{(y')^4}{y^3} + 5y^2y'' + \frac{5}{2}\big[y(y')^2 - y^5\big], \\ y_{01}^2 &= 1/2,\ y_{02} = -y_{01}, \\ y_{03}^2 &= 9/2,\ y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 2, 5),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (-3, 5, 6),\ j = 3, 4. \end{aligned} \tag{3.42}$$

$$\begin{aligned} y^{(4)} &= 5\frac{y'y'''}{y} + \frac{4(4-k^2)}{1-k^2}\frac{(y'')^2}{y} \\ &\quad - \frac{53-17k^2}{1-k^2}\frac{(y')^2y''}{y^2} \\ &\quad + \frac{9(4-k^2)}{1-k^2}\frac{(y')^4}{y^3} + (k^2+11)y^2y'' \\ &\quad + (k^2-19)y(y')^2 + 3(1-k^2)y^5, \\ y_{01}^2 &= 1/(k^2-1),\ y_{02} = -y_{01}, \\ y_{03}^2 &= k^2/(k^2-1),\ y_{04} = -y_{03}, \\ (r_{j1}, r_{j2}, r_{j3}) &= (1, 2, 3),\ j = 1, 2, \\ (r_{j1}, r_{j2}, r_{j3}) &= (6, k, -k),\ j = 3, 4. \end{aligned} \tag{3.43}$$

When $k = 2$, (3.43a) is nothing but the simplified equation

$$\begin{aligned} y^{(4)} &= 5\frac{y'y'''}{y} - 5\Big[\frac{(y')^2}{y^2} - \nu^2y^2\Big]y'' \\ &\quad - 5\nu^2y(y')^2 - \nu^4y^5 + zy + 1. \end{aligned} \tag{3.44}$$

(3.44) was given in [17], and there exists Bäcklund transformation between (3.44) and

$$\begin{aligned} v^{(4)} &= -5v'v'' + 5v^2v'' + 5v(v')^2 \\ &\quad - v^5 + zv + (\nu+1), \end{aligned} \tag{3.45}$$

which was first given in the work of Hone [12], studied in [23, 24] and also proposed as defining a new transcendent [6, 9]. Moreover (3.45) can be obtained as the similarity reduction of the modified Sawada-Kotera (mSK) equation [12].

When $k = 3$, (3.43a) gives the simplified equation

$$\begin{aligned} y^{(4)} &= 5\frac{y'y'''}{y} + \frac{5}{2}\frac{(y'')^2}{y} - \Big[\frac{25}{2}\frac{(y')^2}{y^2} - \frac{5}{2}\nu^2y^2 + \beta\Big]y'' \\ &\quad + \frac{45}{8}\frac{(y')^4}{y^3} - \Big[\frac{5}{4}\nu^2y^2 - \frac{3}{2}\beta\Big]\frac{(y')^2}{y} \\ &\quad - \frac{3}{8}\nu^4y^5 + \frac{1}{2}\beta\nu^2y^3 + zy + 2\epsilon,\ \epsilon = \pm 1. \end{aligned} \tag{3.46}$$

Equation (3.46) was introduced in [17], and there exists Bäcklund transformation between (3.46) and the second member of the generalized second Painlevé equation, P_{II} [6, 9, 21]

$$v^{(4)} = 10v^2v'' + 10v(v')^2 - 6v^5 - \beta(v'' - 2v^3) + zv + \nu. \tag{3.47}$$

(3.47) can be obtained as the similarity reduction of the fifth order modified Korteweg-de Vries (mKdV) equation [12]. Moreover, when $k = 3$, if one lets $y = 1/u$ then (3.43a) gives the simplified equation

$$u^{(4)} = 3\frac{u'u'''}{u} + \frac{7}{2}\frac{(u'')^2}{u} - \frac{17}{2}\frac{(u')^2u''}{u^2} + \frac{27}{8}\frac{(u')^4}{u^3} - \Big(\beta - \frac{5\delta}{u^2}\Big)u'' + \frac{1}{2}\Big(\frac{\beta}{u} - \frac{15\delta}{u^3}\Big)(u')^2 - 2\nu u^2 + 2\alpha z u - \frac{\beta\delta}{u} + \frac{3\delta^2}{2u^3}. \tag{3.48}$$

Equation (3.48) was considered in [20]. When $k = 7$, (3.43a) gives the simplified equation

$$y^{(4)} = 5\frac{y'y'''}{y} + \frac{15}{4}\frac{(y'')^2}{y} - \Big[\frac{65}{4}\frac{(y')^2}{y^2} - \frac{5}{4}\nu^2y^2\Big]y'' + \frac{135}{16}\frac{(y')^4}{y^3} + \frac{5}{8}\nu^2y(y')^2 - \frac{1}{16}\nu^4y^5 + zy - 2. \tag{3.49}$$

Equation (3.49) was introduced in [17], and there exists Bäcklund transformation between (3.49) and (3.45) with $\nu = \hat{\nu} - (3/2)$.

For all the other cases, the simplified equations are the same as the reduced equations (1.3) with the coefficients for $n = -1$.

4. Leading Order $\alpha = -4, -3$

$\alpha = -4, -3$ are also a possible leading order of (1.2) if $F = 0$. For $\alpha = -4$, by adding terms of order -8, we obtain the simplified equation

$$y^{(4)} = a_1\frac{y'y'''}{y} + a_2\frac{(y'')^2}{y} + a_3\frac{(y')^2y''}{y^2} + a_4\frac{(y')^4}{y^3} + dy^5, \tag{4.1}$$

where d is a constant.

Suppose that (1.9), (1.10), (1.11), (1.18), (1.19), (1.20), (1.21), (1.22), (1.34), (1.35) and (1.36) hold, and substitute $y \cong y_0(z - z_0)^{-4} + \beta(z - z_0)^{r-4}$ into (4.1). Then we obtain the following equations for the Fuchs indices r and y_0 [22]:

$$Q(r) = (r + 1)[r^3 + (4a_1 - 23)r^2 - 2(32a_1 + 20a_2 + 8a_3 - 101)r + 4[120a_1 + 100a_2 + 80a_3 + 64a_4 - 210] = 0, \tag{4.2}$$

$$dy_0 + 4(120a_1 + 100a_2 + 80a_3 + 64a_4 - 210) = 0. \tag{4.3}$$

(4.3) implies, that there is only one branch. Now, we determine y_0 and d for each case of (a_1, a_2, a_3, a_4) such that the branch is the principal branch. If $r_0 = -1$ and $r_i, i = 1, 2, 3$ then (4.2) implies that

$$\sum_{i=1}^{3} r_i = -(4a_1 - 23), \quad \sum_{\substack{i,j=1\\ i\neq j}}^{3} r_ir_j = -2(32a_1 + 20a_2 + 8a_3 - 101), \quad \prod_{i=1}^{3} r_i = -4(120a_1 + 100a_2 + 80a_3 + 64a_4 - 210) = dy_0, \tag{4.4}$$

providing that the right hand sides of (4.4) are positive integers.

We have the following equations which have the additional leading terms:

For case 7, (1.21):

$$y^{(4)} = \frac{7}{2}\frac{y'y'''}{y} + 2\frac{(y'')^2}{y} - 8\frac{(y')^2y''}{y^2} + \frac{7}{2}\frac{(y')^4}{y^3} + 24y^2, \quad y_0 = 1, \ (r_1, r_2, r_3) = (2, 3, 4). \tag{4.5}$$

For case 10, (1.35):

$$y^{(4)} = 3\frac{y'y'''}{y} + \frac{9}{4}\frac{(y'')^2}{y} - \frac{29}{4}\frac{(y')^2y''}{y^2} + \frac{49}{16}\frac{(y')^4}{y^3} + 36y^2, \quad y_0 = 1, \ (r_1, r_2, r_3) = (2, 3, 6). \tag{4.6}$$

For all the other cases, $d = 0$ and the simplified equations are the same as the reduced equations (1.3) with the coefficients (a_1, a_2, a_3, a_4), for $n = -4$.

For $\alpha = -3$, the simplified equation is

$$y^{(4)} = a_1\frac{y'y'''}{y} + a_2\frac{(y'')^2}{y} + a_3\frac{(y')^2y''}{y^2} + a_4\frac{(y')^4}{y^3} + fyy', \tag{4.7}$$

where f is a constant. Substituting $y \cong y_0(z-z_0)^{-3} + \beta(z-z_0)^{r-3}$ into (4.7) gives the following equations for the Fuchs indices r and y_0:

$$\begin{aligned} Q(r) = (r+1)[r^3 + (3a_1 - 19)r^2 \\ - 3(13a_1 + 8a_2 + 3a_3 - 46)r \\ + 9[20a_1 + 16a_2 + 12a_3 + 9a_4 - 40] = 0, \end{aligned} \tag{4.8}$$

$$fy_0 - 3(20a_1 + 16a_2 + 12a_3 + 9a_4 - 40) = 0. \tag{4.9}$$

(4.9) implies, that there is only one branch. By following the same procedure, only for case 10, (1.35) we obtain the following equation which has the principal branch and admits the additional leading term (i. e. $f \neq 0$):

$$\begin{aligned} y^{(4)} = \frac{10}{3}\frac{y'y'''}{y} + \frac{8}{3}\frac{(y'')^2}{y} - \frac{82}{9}\frac{(y')^2y''}{y^2} \\ + \frac{112}{27}\frac{(y')^4}{y^3} + 8yy', \\ y_0 = -1, \quad (r_1, r_2, r_3) = (2, 3, 4). \end{aligned} \tag{4.10}$$

For all the other cases, $f = 0$ and the simplified equations are the same as the reduced equations (1.3) with the coefficients (a_1, a_2, a_3, a_4), for $n = -3$.

5. Negative Resonances

In the previous sections, we considered the case of existence of at least one principal branch and obtained the simplified equations. In this section, we present some of the simplified equations which admit the negative resonances. For the leading order $\alpha = -4$, and for case 7, (1.21), case 8, (1.22), case 9, (1.34) and case 10, (1.35), we obtain the following simplified equations:

$$\begin{aligned} y^{(4)} = 5\frac{y'y'''}{y} + 5\frac{(y'')^2}{y} - 17\frac{(y')^2y''}{y^2} + 8\Big[\frac{(y')^4}{y^3} - y^2\Big], \\ y_0 = 21, \ (r_1, r_2, r_3) = (-7, 4, 6), \end{aligned} \tag{5.1}$$

$$\begin{aligned} y^{(4)} = 4\frac{y'y'''}{y} + 3\frac{(y'')^2}{y} - \frac{21}{2}\frac{(y')^2y''}{y^2} + \frac{9}{2}\frac{(y')^4}{y^3} - 4y^2, \\ y_0 = 18, \ (r_1, r_2, r_3) = (-3, 4, 6), \end{aligned} \tag{5.2}$$

$$\begin{aligned} y^{(4)} = 8\frac{y'y'''}{y} + 6\frac{(y'')^2}{y} - 36\frac{(y')^2y''}{y^2} + 24\Big[\frac{(y')^4}{y^3} - y^2\Big], \\ y_0 = 1, \quad (r_1, r_2, r_3) = (-2, -3, -4). \end{aligned} \tag{5.3}$$

$$\begin{aligned} y^{(4)} = 3\frac{y'y'''}{y} + \frac{7}{2}\frac{(y'')^2}{y} - \frac{17}{2}\frac{(y')^2y''}{y^2} + \frac{27}{8}\frac{(y')^4}{y^3} - 8y^2, \\ y_0 = 18, \ (r_1, r_2, r_3) = (-3, 6, 8). \end{aligned} \tag{5.4}$$

The canonical forms of equations (5.2) and (5.4) are also given in [20]. If one lets $u = 1/y$, (5.3) gives $u^{(4)} = 24$.

For the leading order $\alpha = -3$, and for case 4, (1.18), and case 10, (1.35), we obtain the following simplified equations:

$$\begin{aligned} y^{(4)} = 7\frac{y'y'''}{y} + 6\frac{(y'')^2}{y} - 30\frac{(y')^2y''}{y^2} + 18\frac{(y')^4}{y^3} - 6yy', \\ y_0 = 1, \ (r_1, r_2, r_3) = (-3, -2, 3). \end{aligned} \tag{5.5}$$

$$\begin{aligned} y^{(4)} = 2\frac{y'y'''}{y} + 4\frac{(y'')^2}{y} - 2\frac{(y')^2y''}{y^2} + 6yy', \\ y_0 = 20, \ (r_1, r_2, r_3) = (-5, 6, 12). \end{aligned} \tag{5.6}$$

For the leading order $\alpha = -2$, single branch, i.e. $b_3 = 0$, case 2, (1.10), and case 7, (1.21), lead to the following simplified equations:

$$\begin{aligned} y^{(4)} = 6\frac{y'y'''}{y} + 5\frac{(y'')^2}{y} - 22\frac{(y')^2y''}{y^2} + 12\frac{(y')^4}{y^3} - 2yy'', \\ y_0 = 1, \ (r_1, r_2, r_3) = (-2, 2, 3). \end{aligned} \tag{5.7}$$

$$\begin{aligned} y^{(4)} = 5\frac{y'y'''}{y} + 5\frac{(y'')^2}{y} - 17\frac{(y')^2y''}{y^2} + 8\frac{(y')^4}{y^3} \\ - 2[yy'' + (y')^2], \\ y_0 = 1, \quad (r_1, r_2, r_3) = (-2, 2, 5). \end{aligned} \tag{5.8}$$

6. Conclusion

In conclusion, we introduced the simplified equations of non-polynomial fourth order equations with the leading orders $\alpha = -4, -3, -2, -1$, such that all of which pass the Painlevé test. Moreover the compatibility conditions corresponding the parametric zeros; that is, the compatibility conditions at the resonances of the equations obtained by the transformation $y = 1/u$ are identically satisfied. The corresponding simplified equation to (1.2) can be obtained by differentiating twice the leading terms of the third (or fourth) Painlevé equation and adding the terms of order -5 or greater as $z \to z_0$ with constant coefficients such that, $y = 0, \infty$ are the only singular values of equation in y, and they are of order ϵ^{-4} or greater, if one lets $z = \zeta_0 + \epsilon t$. Hence, these equations can be considered as the generalization of the third (or fourth) Painlevé equation.

In the second, third and fourth sections, we investigated the cases of leading order $\alpha = -2, -1$ and $\alpha = -4, -3$, respectively, with the condition of the existence of at least one principal branch. But, in the case of more than one branch, the compatibility conditions at the positive resonances for the second, third and fourth branches are identically satisfied for each case. For the case of $\alpha = -4, -3, -2$ the simplified equations are examined without any restriction. For the case of $\alpha = -1$, $c_1 = 0$, and single and double branch cases were examined without any additional condition. All the other subcases of the case $\alpha = -1$ were investigated for some particular values of n. Some of the equations presented in these sections were considered in the literature before. In the last section, instead of having positive distinct integer resonances, we considered the case of distinct integer resonances. In this case, for the leading order $\alpha = -4, -3$ and $\alpha = -2$ single branch case was considered. Canonical forms of some of the equations given in the last section were introduced in the literature before. The canonical form of all the given simplified equations can be obtained by adding appropriate non-dominant terms with the coefficients analytic in z. The coefficients of the non-dominant terms can be determined from the compatibility conditions at the resonances and from the compatibility conditions corresponding the parametric zeros.

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